GENERALIZED POISSON BRACKETS FOR DISCRETE AND CONTINUOUS SYSTEMS*

APPLICATIONS IN PLASMA PHYSICS

H. Tasso

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Generalized Poisson Brackets are discussed following closely the book of Sudarshan and Mubunda [5] without going into the formalism of symplectic forms. It is found that there are rather drastic differences between discrete and continuous systems, especially with respect to fulfillment of Jacobi identity. Examples are given for the continuous case, and the relation of the brackets to Lagrangian derivation, Liouville theorem and Lie groups are explained. Applications, especially in plasma physics, are mentioned.

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GENERALIZED POISSON BRACKETS FOR DISCRETE AND CONTINUOUS SYSTEMS APPLICATIONS IN PLASMA PHYSICS

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I Introduction

Lagrangian and Hamiltonian formulations of dynamics have had such an impact on physics that the slightest light shed on these formalisms and their applications can be of great interest. In the study of fluids much work has been done since the introduction of Lagrangian and Clebsch [1] variables for the purpose of obtaining Hamiltonian formulations. Usually a canonical Hamiltonian is derived from a standard Lagrangian formulation (see, for example, Goldstein [2]), and canonical Poisson brackets can then be defined from the canonical Hamiltonian equations. Unfortunately, the canonical variables sometimes turn out to be artificial (potentials) and do not correspond to the desired physical representation. This can be overcome by using non-canonical transformations as in Ref. [3], which allow non-canonical Poisson brackets to be defined in physical variables (see also an example from Clebsch in Ref. [4] for discrete systems).

The purpose of this contribution is to deal with the question of defining Poisson brackets directly in the desired variables independently of the fact whether or how they are related to some canonical variables. Poisson brackets of this type are called here Generalized Poisson Brackets (GPB) according to the nomenclature of Sudarshan and Mukunda [5]. This reference deals essentially with the discrete case and contains very interesting material, which is presented here in connection with the continuous case, so that we begin with GPB of discrete systems. GPB of continuous systems with simple applications are treated in Sec. III, which is followed by the conclusion.

II GPB for Discrete Systems [5]

Let us consider N real quantities z^μ, $\mu = 1 \ldots N$ and 2 arbitrary functions of the vector z f(z), g(z) and define

$$[f,g] = \eta^{\mu\nu}(z) \frac{\partial f}{\partial z^\mu} \frac{\partial g}{\partial z^\nu} , \tag{1}$$

where $\eta^{\mu\nu}$ is an antisymmetric matrix and the repetition of indices means summation for their values from 1 to N.

To prove that expression (1) has the main properties of a Poisson bracket, we have to check four properties, viz. antisymmetry, linearity, product rule and Jacobi identity:

1) Antisymmetry is obvious $[f,g] = -[g,f]$. (2)

2) Linearity with respect to f is easily seen

$$[c_1 f_1 + c_2 f_2, g] = c_1 [f_1, g] + c_2 [f_2, g] . \tag{3}$$

3) The product rule is

$$[f_1 f_2, g] = [f_1, g] f_2 + f_1 [f_2, g] \tag{4}$$

and follows from the properties of the first derivative.

4) Jacobi identity is the non-trivial property to check and, as we shall see, it puts restrictions on the $\eta^{\mu\nu}$ as functions of z.

$$[[f,g], h] + [[h,f],g] + [[g,h], f] = 0. \tag{5}$$

Using expression (1) of the GPB, we can write the left-hand side of identity (5) in the following form:

$$\eta^{\rho\lambda} \frac{\partial}{\partial z^{\rho}} \left(\eta^{\mu\nu} \frac{\partial f}{\partial z^{\mu}} \frac{\partial g}{\partial z^{\nu}} \right) \frac{\partial h}{\partial z^{\lambda}} + \eta^{\rho\nu} \frac{\partial}{\partial z^{\rho}} \left(\eta^{\lambda\mu} \frac{\partial h}{\partial z^{\lambda}} \frac{\partial f}{\partial z^{\mu}} \right) \frac{\partial g}{\partial z^{\nu}}$$

$$+ \eta^{\rho\mu} \frac{\partial}{\partial z^{\rho}} \left(\eta^{\nu\lambda} \frac{\partial g}{\partial z^{\nu}} \frac{\partial h}{\partial z^{\lambda}} \right) \frac{\partial f}{\partial z^{\mu}} .$$

All the indices are dummy and some of them have been cyclically permuted for the purpose of the proof. We can group the terms into those containing derivatives of the $\eta^{\mu\nu}$ and those without derivatives of the $\eta^{\mu\nu}$, so that we obtain

$$\left(\eta^{\rho\lambda} \frac{\partial \eta^{\mu\nu}}{\partial z^{\rho}} + \eta^{\rho\nu} \frac{\partial \eta^{\lambda\mu}}{\partial z^{\rho}} + \eta^{\rho\mu} \frac{\partial \eta^{\nu\lambda}}{\partial z^{\rho}} \right) \frac{\partial f}{\partial z^{\mu}} \frac{\partial g}{\partial z^{\nu}} \frac{\partial h}{\partial z^{\lambda}} +$$

$$\left(\eta^{\rho\lambda} \eta^{\mu\nu} + \eta^{\rho\nu} \eta^{\lambda\mu} \right) \frac{\partial^2 f}{\partial z^{\rho} \partial z^{\mu}} \frac{\partial g}{\partial z^{\nu}} \frac{\partial h}{\partial z^{\lambda}} +$$

$$\left(\eta^{\rho\lambda} \eta^{\mu\nu} + \eta^{\rho\mu} \eta^{\nu\lambda} \right) \frac{\partial f}{\partial z^{\mu}} \frac{\partial^2 g}{\partial z^{\rho} \partial z^{\nu}} \frac{\partial h}{\partial z^{\lambda}} +$$

$$\left(\eta^{\rho\nu} \eta^{\lambda\mu} + \eta^{\rho\mu} \eta^{\nu\lambda} \right) \frac{\partial f}{\partial z^{\mu}} \frac{\partial g}{\partial z^{\nu}} \frac{\partial^2 h}{\partial z^{\rho} \partial z^{\lambda}} . \tag{6}$$

If we exchange the indices of the second derivatives in expression (6), the corresponding parentheses change sign owing to the antisymmetry of the $\eta^{\mu\nu}$, so that all these terms vanish identically. We want the identity to be true for all f, g, h and we end by requiring that every coefficient of $\frac{\partial f}{\partial z^{\mu}} \frac{\partial g}{\partial z^{\nu}} \frac{\partial h}{\partial z^{\lambda}}$ vanish, i.e.

$$\eta^{\rho\lambda} \frac{\partial \eta^{\mu\nu}}{\partial z^{\rho}} + \eta^{\rho\nu} \frac{\partial \eta^{\lambda\mu}}{\partial z^{\rho}} + \eta^{\rho\mu} \frac{\partial \eta^{\nu\lambda}}{\partial z^{\rho}} = 0, \text{ all } \lambda, \mu, \nu. \tag{7}$$

Jacobi's identity (5) is then equivalent to condition (7).

Relation (7) is obviously verified for constant $\eta^{\mu\nu}$, which includes the canonical case. It can easily be checked, in general, for small systems but is not convenient for finding the general form of the $\eta^{\mu\nu}$. As we shall see, this can be answered for the inverse matrix

$\eta_{\mu\nu}$ if it exists. A necessary condition for the existence of the $\eta_{\mu\nu}$ is that N is even. Assuming the existence of the $\eta_{\mu\nu}$, we have

$$\eta_{\mu\nu}\,\eta^{\nu\lambda} = \delta^{\lambda}_{\mu} \tag{8}$$

where δ^{λ}_{μ} is the Kronecker symbol. Now multiplying relation (7) first by $\eta_{\alpha\lambda}$, then by $\eta_{\beta\mu}$ and then by $\eta_{\gamma\nu}$ and using at every stage relation (8) and its derivatives, we obtain

$$\frac{\partial\eta_{\beta\gamma}}{\partial z^{\alpha}} + \frac{\partial\eta_{\alpha\beta}}{\partial z^{\gamma}} + \frac{\partial\eta_{\gamma\alpha}}{\partial z^{\beta}} = 0. \tag{9}$$

The calculations can be reversed in a similar way to go back to relation (7), so that relations (7) and (9) are identical.

The advantage of relation (9) is that it can be solved by

$$\eta_{\mu\nu} = \frac{\partial A_{\nu}}{\partial z^{\mu}} - \frac{\partial A_{\mu}}{\partial z^{\nu}}, \tag{10}$$

where the A_{μ} are a general set of N functions. It is, of course, not easy in practice to calculate the $\eta^{\mu\nu}$ for large N because one has to take the inverse of a large matrix.

It may be interesting to note how GPB (1) is related to non-standard Lagrangians:

$$\mathcal{L} = \int_a^b L\,dt = \int_a^b \Big[\sum_{s=1}^{N} A_s(q)\,\dot{q}^s - V(q)\Big]\,dt. \tag{11}$$

The Euler - Lagrange equations read

$$\sum_{r=1}^{N} \eta_{sr}\,\dot{q}^r = \frac{\partial V}{\partial q_s}, \tag{12}$$

with η_{sr} given by relation (10). If the matrix of the η_{sr} is non-singular, then η^{sr} can be defined and will automatically satisfy relation (7) and $\left[f(q), g(q)\right] = \eta^{rs} \frac{\partial f}{\partial q^r} \frac{\partial g}{\partial q^s}$ is a GPB. If we apply the matrix of the η^{sr} to equation (12), we obtain

$$\dot{q}^s = \sum_{r=1}^{N} \eta^{sr} \frac{\partial V}{\partial q^r} = \left[q^s, V\right] . \tag{13}$$

Singular GPB

All that has been said until relation (7) is true even of matrices η^{ij} which are singular. If for such a case the rank of the matrix η^{ij} is N-m (which has to be even), then it is possible to show [5] that a N × N matrix $\eta'^{\mu\nu}$ can be constructed so that

$$\eta'^{\mu\nu} = \left(\begin{array}{c|c} \eta'^{rs} & 0 \\ \hline 0 & 0 \end{array}\right) \begin{array}{l} \updownarrow N-m \\ \updownarrow m \end{array}$$

and the matrix η'^{rs} is nonsingular. This is achieved by a change of variable which need not be simple nor is it always useful because many properties of the original singular matrix may be lost.

A final remark concerns the Darboux theorem, which gives the possibility of always finding a canonical representation locally in phase space so that any GPB could be transformed in principle into a canonical Poisson bracket. Again this does not need to be the best strategy for all problems, and examples will be given later in which the GPB is much simpler than the standard Poisson bracket.

Dirac Brackets [6]

For quantizing nonlinear electrodynamics and the gravitational field, non-standard Lagrangians with the property

$$\left| \frac{\partial^2 L}{\partial \dot{q}^i \, \partial \dot{q}^j} \right| = 0 \tag{14}$$

have to be faced. Condition (14) leads to constraints for the canonical variables p, q and necessitates generalization of the standard Hamiltonian dynamics. The theory with constraints is rather complicated and can be found in Refs. [5] and [6]. It turns out that the Poisson brackets thus introduced are singular GPB. A detailed discussion [6] of the primary and secondary constraints helps to find out the restricted variables with respect to which the GPB is nonsingular.

Liouville Theorem

A canonical flow in phase space conserves the volume $\prod_i d\,p_i\; d\,q_i$. In the case of flows of the type of eq. (13) the property holds [7] if the η^{ij} are constant, which can be seen as follows:

$$\frac{\partial\, \dot{q}^s}{\partial\, q^s} = \eta^{sr} \frac{\partial^2 V}{\partial q^r \partial q^s} = 0 \tag{15}$$

the second equality being due to the antisymmetry of η^{sr}.

If the η^{sr} are not constant, the flow becomes compressible and

$$\prod_i d\, q^i \neq \text{ct.}$$

It is, however, possible to change coordinates by the Darboux theorem and return to a canonical system, but the Jacobian is not easy to obtain explicitly for large systems.

Another drawback for the case of non-constant η^{ij} is that the Liouville operator is no longer Hermitian and many problems in statistical mechanics become much harder. Recent developments [8] concerning non-equilibrium entropy strongly depend upon hermiticity.

Poisson's Theorem

It should be noted that the GPB of two constants of motion is also a constant of motion. The proof makes use of the antisymmetry and the Jacobi identity of the GPB.

III GPB for Continuous Systems

The representation of the system now consists of a set of real functions u_i having differentiability and integrability properties enough to carry through the formalism. Instead of a sum over the different real variables we have a sum over the different functions and we have integrations over the values of the functions. The observables are no longer functions of the phase space real variables but functional of the real functions, integrals to be more precise. If F and G are two such functionals, then the expected GPB is

$$[F,G] = \sum_{i,j=1}^{N} \int \frac{\delta F}{\delta u_i} A_{ij} \frac{\delta G}{\delta u_j} dx \quad , \tag{16}$$

where the A_{ij} are antisymmetric operators and $A_{ij} = A_{ji}$.

The Lie algebra properties of antisymmetry, linearity and product rule are easy to see, but the Jacobi identity is not trivial, as we shall see.

The reason is that we have to take the functional derivative of $[F,G]$ to be able to construct $[[F,G], H]$. The derivative of $[F,G]$ involves the derivatives of nonlinear operators, especially of the A_{ij} with respect to their dependence upon the u'_s. Let us first say a few words about such derivatives, called Fréchet derivatives.

The Fréchet derivative [9] N'_u of a nonlinear operator N is defined as follows:

$$\text{norm}\left(\frac{d}{d\varepsilon} N(u+\varepsilon w)\Big|_{\varepsilon=0} - (N'_u\, w)\right) = 0 \qquad (17)$$

In the case of a functional $F = \int f(u,u_x \ldots)\, dx$, which can be considered as a nonlinear integral operator, we have

$$\frac{d}{d\varepsilon} F(u+\varepsilon w)\Big|_{\varepsilon=0} = \int \frac{\delta F}{\delta u}\, w\, dx \quad , \qquad (18)$$

the functional derivative being $\dfrac{\delta F}{\delta u} = \dfrac{\partial f}{\partial u} + \dfrac{d}{dx}\dfrac{\partial f}{\partial u_x} + \ldots$ (19)

In this case it is rather easy to isolate w inside the norm and obtain explicitly the Fréchet derivative. Generally, the Fréchet derivative of differential operators should be obtained by isolating w by integrations by parts. Higher-order Fréchet derivatives can also be constructed and the order of derivations can be exchanged. It can thus be stated (see Ref. [9]) that the second derivative of a functional is a symmetric operator, which will be used extensively below.

Let us now calculate the functional derivative of $[F,G]$ of expression (16) and start with the intermediate calculation

$$\frac{d}{d\varepsilon} \boxed{F,G}(u_k + \varepsilon w)\Big|_{\varepsilon=0} = \int \left(\frac{\delta F}{\delta u_i}\right)'_{u_k} w\, A_{ij} \frac{\delta G}{\delta u_j}\, d\underline{x}$$

$$+ \int \frac{\delta F}{\delta u_i} A_{ij} \left(\frac{\delta G}{\delta u_j}\right)'_{u_k} w\, d\underline{x}$$

$$+ \int \frac{\delta F}{\delta u_i} \frac{\delta A_{ij}}{\delta u_k}(w) \frac{\delta G}{\delta u_j}\, d\underline{x} \qquad (20)$$

To obtain the functional derivative, we have to isolate w under the integral. This is easy for the first two terms because $\frac{\delta^2 F}{\delta u_i \delta u_k}$ and $\frac{\delta^2 G}{\delta u_i \delta u_k}$ are symmetric operators. The third term is very difficult to treat for general operators A_{ij}. For a specific differential operator successive appropriate integrations by parts can completely isolate w. We already see at this stage that the situation is totally different from the discrete case, where the derivatives of the $\eta^{\mu\nu}$ of expression (1) are straightforward. Let us assume that we have isolated w in the third term, and that it can be written as $B^k_{ij}\left(\frac{\delta F}{\delta u_i}, \frac{\delta G}{\delta u_j}\right)$. The functional derivative of $\boxed{F,G}$ is now

$$\frac{\delta \boxed{F,G}}{\delta u_k} = \frac{\delta^2 F}{\delta u_i \delta u_k} A_{ij} \frac{\delta G}{\delta u_j} - \frac{\delta^2 G}{\delta u_j \delta u_k} A_{ij} \frac{\delta F}{\delta u_i}$$

$$+ B^k_{ij}\left(\frac{\delta F}{\delta u_i}, \frac{\delta G}{\delta u_j}\right) \qquad (21)$$

If the A_{ij} do not depend upon the u's, then $B^k_{ij} = 0$. Using eq.(21), we can verify Jacobi identity for that case [10] and discover that the symmetry of the Fréchet second derivatives and the antisymmetry of the A_{ij} are sufficient to satisfy Jacobi identity. This corresponds to $\eta^{\mu\nu}$ = ct. in the discrete case, for which condition (7) is identically verified. If the $B^k_{ij} \neq 0$, then the Jacobi identity reduces to

$$\left[\left[F,G\right], H\right] + \left[\left[H, F\right], G\right] + \left[\left[G,H\right], F\right] =$$

$$\int B^k_{ij} \left(\frac{\delta F}{\delta u_i}, \frac{\delta G}{\delta u_j}\right) A_{kl} \frac{\delta H}{\delta u_l} d\underline{x} + \int B^k_{ij} \left(\frac{\delta H}{\delta u_i}, \frac{\delta F}{\delta u_j}\right) A_{kl} \frac{\delta G}{\delta u_l} d\underline{x}$$

$$+ \int B^k_{ij} \left(\frac{\delta G}{\delta u_i}, \frac{\delta H}{\delta u_j}\right) A_{kl} \frac{\delta F}{\delta u_l} d\underline{x} = 0 . \qquad (22)$$

Condition (22) is analogous to condition (7) with the important difference that condition (7) is completely explicit and is identical to condition (9) if the matrix $\eta^{\mu\nu}$ is non-singular and condition (9) can be completely solved by relation (10). In the continuous case condition (22), which must be fulfilled for all functionals F, G, H, does not permit explicit determination of the operators A_{ij} because the functional derivatives of F, G and H cannot be isolated under the integrals. Moreover, the analogon of condition (9) is missing so that there is a real drawback in the continuous case. In practice one guesses a GPB and tries to verify Jacobi identity. An explicit condition can be obtained [11], however, if the A_{ij} depend linearly upon the u's.

This is now the right place to see how this formalism works when applied to partial differential equations occuring in physics.

Example 1: Korteweg de Vries Equation

This equation is of the form

$$u_t = uu_x + u_{xxx} . \qquad (23)$$

The adequate GPB in this case [12] is

$$\left[F,G\right] = \int_{-\infty}^{+\infty} \frac{\delta F}{\delta u} \frac{\partial}{\partial x} \frac{\delta G}{\delta u} dx. \qquad (24)$$

There is only one antisymmetric operator $A = \frac{\partial}{\partial x}$ which is independent of u so that Jacobi identity is verified. Equation (23) itself can be written in terms of the GPB of u and the Hamiltonian

$$H = \int_{-\infty}^{+\infty} \frac{u^3}{6}\, dx - \int_{-\infty}^{+\infty} \frac{u^2 x}{2}\, dx \tag{25}$$

as $$u_t = [u,H] . \tag{26}$$

(Note that $u = \int u(x')\delta(x'-x)\, dx'$ and $\frac{\delta u}{\delta u(x')} = \delta(x-x')$) .

The advantage of GPB (24) is essentially that it simplifies calculations, and that it is written directly in the desired variable. An interesting application is to understand the relation between the constants of the motion of equation (23). At this point we can use Poisson's theorem and try to derive new constants of motion from those already known by calculating their GPB. It is well known and can easily be checked that

$$G = \int (xu + t\, \frac{u^2}{2})\, dx \tag{27}$$

is a constant of motion (stating that the centre of mass has a constant velocity). Taking now any functional defined by

$$F = \int T(u,\, u_x \ldots u_n,\, x,t)\, dx\ , \tag{28}$$

we find [13] after integrations by parts that

$$[F,G] = \int \frac{\delta F}{\delta u}\, dx = \int \frac{\partial T}{\partial u}\, dx\ . \tag{29}$$

If F is one of the constants of motion of eq. (23) with T polynomial, we obtain a relation between successive constants of motion through polynomials of different degrees:

$$\frac{\partial T_r}{\partial u} = T_{r-1} . \tag{30}$$

This relation is proved with much more effort in the literature on Korteweg de Vries equation.

Example 2: Two-dimensional vorticity equation and guiding centre plasmas

The equations are

$$\omega_t = - \underline{v} \cdot \nabla \omega , \tag{31}$$

$$\nabla \cdot \underline{v} = 0 , \tag{32}$$

with $\omega = \underline{e}_z \cdot \nabla \times \underline{v}$. (33)

One can express $\underline{v}$ in terms of ω using eq. (32) and the Green's function, which inverts the Laplace operator. One obtains

$$\omega_t = - \int \omega(\underline{x}') \, \underline{M}(\underline{x}|\underline{x}') \, d\tau . \nabla\omega, \tag{34}$$

with $\underline{M} = \underline{e}_z \times \nabla \, k(\underline{x}|\underline{x}')$, (35)

and k is the Green's function of the Laplacian in 2 dimensions for an infinite medium.

If we take [14] as GPB

$$[F,G] = \int \omega(\underline{x}) \, \{\frac{\delta F}{\delta \omega} , \frac{\delta G}{\delta \omega}\} \, d\tau , \tag{36}$$

where $$\{f,g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} , \qquad (37)$$

then $$\frac{\delta [F,G]}{\delta\omega} = \omega\{\frac{\delta F}{\delta\omega} , \frac{\delta G}{\delta\omega}\}'_{\omega} + \{\frac{\delta F}{\delta\omega} , \frac{\delta G}{\delta\omega}\} . \qquad (38)$$

Using eqs. (36) and (38) in Jacobi's identity leads after rather lengthy calculations to its verification.

GPB and Lie Groups

As is to be expected, a certain number of constants of motion are related to Lie group properties, more precisely to external Lie groups. The GPB of these constants of motion with the dynamical variables of the system produce the generators of the group. In the case of the Korteweg-de Vries equation these constants are

$$P = \int \frac{u^2}{2} dx, \qquad (39)$$

$$H = \int \frac{u^3}{6} dx - \int \frac{u_x^2}{2} dx , \qquad (40)$$

$$G = \int (x u + t \frac{u^2}{2}) dx . \qquad (41)$$

The constants (39), (40) and (41) correspond, respectively, to momentum, energy and uniform velocity of the centre of mass. The GPB of P, H and

G with u are

$$[P, u] = - \frac{\partial u}{\partial x} , \tag{42}$$

$$[H, u] = - \frac{\partial u}{\partial t} , \tag{43}$$

$$[G, u] = -(1 + t\frac{\partial u}{\partial x}) . \tag{44}$$

Equations (42), (43) and (44) completely define the three generators of the one-dimensional Galilei group.

This simple example illustrates the relation between the GPB and the Lie algebra of the underlying group of the dynamical system. This can help to find the right guess for the GPB. It is indeed well-known that the generators of the Lie group verify Jacobi identity. The next step is to establish a correspondence [15] between the generators and the dynamical variables and derive the correspondence between the commutators and the GPB. This is very much exploited in, for example, Ref. [15].

IV Survey

Generalized Poisson Brackets for discrete systems are not only a chapter of classical mechanics or a pedestrian way of working with symplectic forms [16], they are also needed in many cases such as for non-standard Lagrangians of Field theory in the form of Dirac brackets or when approximations within classical mechanics are done so that the problem becomes non-standard. An important application [17] recently appeared in plasma physics for the drift equations of a particle gyrating in a magnetic field.

Several applications for ideal fluids and fields have appeared in the literature [3,14,15]. Especially in plasma physics [14], many forms of GPB have recently appeared for the ideal MHD equations, for guiding centre plasmas and for the Vlasov-Maxwell system. The main advantage lies in the Hamiltonian formulation of continuous systems in terms of non-canonical laboratory variables.

Many theorems of dynamics only depend upon the Lie algebra structure of the GPB, and for some problems non-canonical variables may be easier to handle. On the other hand, in the discrete case the inversion of large matrices is a real problem, and in the continuum case one has to check Jacobi identity, which can be very tedious. In this respect one could use symbolic computation [18] programs which the author has not yet seen applied for this purpose in the literature.

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